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A Constructive Approach to Minimal Projections in Banach Spaces

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Let X be a Banach space and Y a finite-dimensional subspace of X. Let P be a minimal projection of X onto Y. It is shown (Theorem 1.1) that under certain conditions there exist sequences of finite-dimensional "approximating subspaces" X_m and Y_m of X with corresponding minimal projections P_m: $X_m \to Y_m$, such that $\lim_{m\to\infty} \|P_m\| = \|P\|$. Moreover, a certain related sequence of projections $\imath_m \circ P_m \circ \pi_m$: $X \to Y$ has cluster points in the strong operator topology, each of which is a minimal projection of X onto Y. When $X = C[a, b]$ the result reduces to a theorem of Cheney and Morris ("The Numerical Determination of Projection Constants," Report No. 75, Center for Numerical Analysis, The University of Texas at Austin, 1973). It is shown (Corollary 1.11) that the hypothesis of Theorem 1.1 holds in many important Banach spaces, including $C[a, b]$, $L^p[a, b]$ and l^p for $1 \leqslant p < \infty$, and c_0, the space of sequences converging to zero in the sup norm.

0. PRELIMINARIES

Let X be a Banach space and let Y be a fixed subspace of X. A linear operator P: $X \to Y$ will be called a projection of X onto Y if $Py = y$, for all $y \in Y$.

A projection P will be called a *minimal projection* of X onto Y if the operator norm of P is less than or equal to the operator norm of any other projection Q from X onto Y. That is, P is a minimal projection of X onto Y if

$$\|P\| \leqslant \|Q\| \qquad \text{for all} \quad Q \in \mathscr{P}(X, Y),$$

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where

$$\|T\| = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\|,$$

and $\mathscr{P}(X, Y) = \{Q : Q \text{ is a projection from } X \text{ onto } Y\}$. Similarly, a co-minimal projection from X onto Y is any projection $P_c \in \mathscr{P}(X, Y)$ such that

$$\|I - P_c\| \leqslant \|I - Q\| \qquad \text{for all} \quad Q \in \mathscr{P}(X, Y),$$

where $I: X \to X$ is the identity operator.

In what follows we will often use the fact that if Y is a finite-dimensional subspace of X, then a minimal projection from X onto Y always exists (Morris and Cheney [15]). On the other hand, minimal projections are not always unique. No uniqueness results will be assumed or needed in this paper.

It should also be mentioned that in some important cases minimal and co-minimal projections coincide. For example, Daugavet [8] has shown that for a compact operator T on $C[a, b]$ (the continuous real-valued functions on the interval $[a, b]$) we have

$$\|I - T\| = 1 + \|T\|. \tag{1}$$

If Y is finite dimensional, Eq. (1) implies that minimal and co-minimal projections are the same. Recently, Babenko and Pichugov [1] have shown that (1) holds for the compact operators on $L^1(0, 1)$. (Incidentally, this last result shows that the minimal projection of L^1 onto the span of 1 and x given in [9] is also co-minimal.)

In addition to the notation defined above we introduce the following: X, Y, Z and W are always Banach spaces and Y is often a finite-dimensional subspace of X. The identity map is denoted by I. Define the boundary of the unit ball in X, $\partial U(X) = \{x \in X: \|x\| = 1\}$. The set of all bounded linear operators from a Banach space X into a Banach space Y will be denoted by $\mathscr{B}(X, Y)$. The notation $\mathscr{B}(X, X)$ is shortened to $\mathscr{B}(X)$. No spcial subscripting on the norm symbol will be used to distinguish the norms in various spaces or the restriction of a norm to a subspace. Definitions and terminology relating to nets will be those found in Kelly [14]. If Q is a set, $|Q|$ is the cardinality of Q, and ϕ will denote the empty set. The natural numbers will be denoted by N.

1. Generalized Discretization in Banach Spaces

In this section we develop a technique for computing minimal projections and show that it is applicable to many important Banach spaces, including $C[a, b]$, $L^p[a, b]$ and l^p for $1 \leqslant p < \infty$, and c_0, the space of sequences convergent to zero in the sup norm. This technique may be roughly described as a generalized process of discretization. Given a minimal projection $P: X \to Y$ of X onto Y, the idea is to construct a sequence of "approximating subspaces" X_m and Y_m, with corresponding minimal projections $P_m: X_m \to Y_m$, in such a way that the sequence $\{P_m\}$ "converges" to P. An exact sense in which this construction may be carried out is contained in Theorem 1.1. The structure of the theorem is motivated by the following example. Let $X = C[a, b]$, the continuous real-valued functions on $[a, b]$, with Y a finite-dimensional subspace of X, and let $\{Q_m : m \in N\}$ be a sequence of finite point-sets of $[a, b]$ such that $\bigcup Q_m$ is dense in $[a, b]$. Let π_m: $C[a, b] \to C[a, b]$ be the map defined by "piecewise-linear interpolation" at the points of Q_m. That is, define $\pi_\pi f$ in $[t_i, t_{i+1}]$ by

$$(\pi_m f)(t) = \frac{f(t_i)(t_{i+1} - t)}{t_{i+1} - t_i} + \frac{f(t_{i+1})(t - t_i)}{t_{i+1} - t_i}.$$

We may then define $X_m = \pi_m(X)$, and $Y_m = \pi_m(Y)$. Thus, when $X = C[a, b]$ a sequence of finite-dimensional approximating subspaces X_m and Y_m arises, each consisting of piecewise-linear continuous functions on $[a, b]$. It seems natural to try to gain information about P from the sequence of minimal projections $P_m: X_m \to Y_m$. This is what is done in the theorem with the specifics of this example replaced by more general conditions. We state the theorem first, deferring the proof until several lemmas have been established.

Theorem 1.1. *Let X be an arbitrary real or complex Banach space, with Y an n-dimensional subspace of X and P a minimal projection from X onto Y. For each $m \in N$ let π_m: $X \to X$*

$$\begin{array}{ccc} X & \xrightarrow{\;P\;} & Y \\ {\scriptstyle \pi_m}\downarrow & & \uparrow{\scriptstyle (\pi_m | Y)^{-1}} \\ X_m & \xrightarrow[P_m]{} & Y_m \end{array} \qquad \text{(not commutative)}$$

be a norm 1 projection of X onto $X_m = \pi_m(X)$ such that $\pi_m x \to x$ as $m \to \infty$ for each fixed $x \in X$, and P_m a minimal projection of X_m onto $Y_m = \pi_m(Y)$. Then for all sufficiently large m, $(\pi_m | Y)^{-1}$ exists and

(a) $\|(\pi_m | Y)^{-1} \circ P_m \circ \pi_m\| \to \|P\|$ *and* $\|P_m\| \to \|P\|$ *as* $m \to \infty$;

(b) *the sequence of projections* $\{(\pi_m | Y)^{-1} \circ P_m \circ \pi_m\}$ *of* X *onto* Y *has cluster points in the strong operator topology and each of these is a minimal projection of* X *onto* Y. *If* X *is separable, then some subsequence converges in the strong operator topology to a minimal projection of* X *onto* Y.

Remark. Both X_m and Y_m are Banach spaces with the norm inherited from X, but for different reasons. The range of a bounded projection is a closed subspace, so X_m is a Banach space. As for Y_m, the linear map $\pi_m | Y$ cannot increase dimension and so Y_m is a finite-dimensional subspace of X, and hence a Banach space. Note that $\pi_m | Y$ is *not*, in general, a projection from Y onto Y_m. Generally speaking, Y_m is not a subset of Y.

PROPOSITION 1.2. *Let* Z *and* W *be Banach spaces with* Z *finite dimensional. For each* $m \in N$ *let* $T_m \in \mathscr{B}(Z, W)$. *If* $\lim_{m \to \infty} \|T_m z\| = 0$ *for all* $z \in Z$, *then* $\lim_{m \to \infty} \|T_m\| = 0$.

Proof. By the principle of uniform boundedness $\|T_m\| < M$ for all m. Since $\partial U(Z)$ is compact, given $\varepsilon > 0$ there are $z_1, ..., z_k \in \partial U(Z)$ such that for any $z \in \partial U(Z)$ we have $\|z - z_j\| < \varepsilon/2M$ for some $1 \leqslant j \leqslant k$. There exists n_0 such that if $m \geqslant n_0$, $\|T_m z_i\| < \varepsilon/2$ for $i = 1, ..., k$. Therefore if $m \geqslant n_0$ and $z \in \partial U(Z)$, then

$$\|T_m z\| \leqslant \|T_m(z - z_j)\| + \|T_m z_j\| \leqslant \|T_m\| \, \|z - z_j\| + \varepsilon/2 < \varepsilon.$$

PROPOSITION 1.3. *Let* W *be a Banach space and for each* $m \in N$ *let* $T_m \in \mathscr{B}(W)$ *be such that* $\|T_m - I\| \to 0$ *as* $m \to \infty$. *Then for all sufficiently large* m, T_m^{-1} *exists and* $\|T_m^{-1} - I\| \to 0$.

Proof. Let $S_m = I - T_m$. Given $0 < \varepsilon < 1$ we can choose m large so that that $\|S_m\| < \varepsilon$. By a well-known result,

$$T_m^{-1} = (I - S_m)^{-1} = I + S_m + S_m^2 + S_m^3 + \cdots.$$

That is, $T_m^{-1} - I = S_m + S_m^2 + S_m^3 + \cdots$, where

$$\|T_m^{-1} - I\| \leqslant \sum_{k=1}^{\infty} \|S_m^k\| \leqslant \sum_{k=1}^{\infty} \|S_m\|^k = \frac{1}{1-\varepsilon} - 1.$$

Therefore, $\|T_m^{-1} - I\| \to 0$ as $m \to \infty$.

LEMMA 1.4. *Referring to Theorem* 1.1, *each of the following holds for all sufficiently large* m:

(a) *$P | Y_m$ is invertible;*

(b) *$\pi_m | Y$ is invertible.*

Moreover,

(c) $\|(\pi_m | Y)^{-1} \circ (P | Y_m)^{-1}\| \to 1$, *as* $m \to \infty$.

Proof. By hypothesis, for each $y \in Y$, $(\pi_m | Y)(y) \to y$ as $m \to \infty$. Since P is continuous $P((\pi_m | Y)(y)) \to Py = y$ as $m \to \infty$. Therefore, for all $y \in Y$,

$$\|((P | Y_m) \circ (\pi_m | Y)) y - (I | Y) y\| \to 0 \qquad \text{as} \quad m \to \infty.$$

Note that

$$[(P | Y_m) \circ (\pi_m | Y) - I | Y] : Y \to Y.$$

Therefore, by Proposition 1.2 with $Z = Y$ and $W = Y$,

$$\|(P | Y_m) \circ (\pi_m | Y) - I | Y\| \to 0 \qquad \text{as} \quad m \to \infty.$$

Since $[(P | Y_m) \circ (\pi_m | Y)] \in \mathscr{B}(Y)$, by Proposition 1.3 $[(P | Y_m) \circ (\pi_m | Y)]^{-1}$ exists for all sufficiently large m and

$$\|[(P | Y_m) \circ (\pi_m | Y)]^{-1} - I | Y\| \to 0 \qquad \text{as} \quad m \to \infty. \tag{2}$$

Therefore, $[(P | Y_m) \circ (\pi_m | Y)]$ is one-to-one and onto. By a well-known result $\pi_m | Y : Y \to Y_m$ must be one-to-one and $P | Y_m : Y_m \to Y$ must be onto. Since Y and Y_m are finite-dimensional spaces it follows that $\pi_m | Y$ and $P | Y_m$ are both invertible. Therefore, $[(P | Y_m) \circ (\pi_m | Y)]^{-1} = (\pi_m | Y)^{-1} \circ (P | Y_m)^{-1}$ implies that $\|(\pi_m | Y)^{-1} \circ (P | Y_m)^{-1}\| \to 1$ as $m \to \infty$.

DEFINITION 1.5. Referring to Theorem 1.1, we define $\imath_m : Y_m \to Y$ by $\imath_m = (\pi_m | Y)^{-1}$ and assume without loss of generality that the π_m have been re-indexed if necessary so that $\imath_m$ is defined for $m = 1, 2, \ldots$.

The next few pages will be devoted to showing that $\|\imath_m\| \to 1$ as $m \to \infty$. It might seem that this should be immediate by:

(a) $\|(\pi_m | Y) y - y\| \to 0$ as $m \to \infty$. Then by Proposition 1.2 we have

(b) $\|\pi_m | Y - I | Y\| \to 0$ as $m \to \infty$. By Proposition 1.3 it follows that

(c) $\|(\pi_m | Y)^{-1} - I | Y\| \to 0$ as $m \to \infty$, i.e., $\|\imath_m - I | Y\| \to 0$ as $m \to \infty$ so that $\|\imath_m\| \to 1$ as $m \to \infty$.

The problem is that (c) does not follow from (b) by Proposition 1.3, because the range of $\pi_m | Y$ is not, in general, a subset of Y. The proof of Proposition 1.3 relies heavily on this fact. Moreover, there does not seem to be any way of making this argument work by extending the definition of the maps involved or by extending ranges or domains.

Before continuing, some definitions are needed.

DEFINITION 1.6. A linear operator $T \in \mathscr{B}(Z, W)$ is said to be *bounded below* if there exists a constant $M > 0$ such that $\|Tz\| \geqslant M$ for all $z \in \partial U(Z)$. If T is bounded below we denote the greatest lower bound of T by $\operatorname{glb}(T)$.

PROPOSITION 1.7. *Let Z and W be arbitrary Banach spaces with $T \in \mathscr{B}(Z, W)$. If T is one-to-one and Z is finite dimensional, then T is bounded below.*

Proof. Since Z is finite dimensional, $\partial U(Z)$ is compact and so the continuous function $\|Tz\|$ assumesits infimum at some point $z_0 \in \partial U(Z)$, i.e.,

$$\operatorname{glb}(T) = \|Tz_0\|, \qquad \|z_0\| = 1.$$

Now T is one-to-one so $\|Tz_0\| \neq 0$.

PROPOSITION 1.8. *Let Z and W be arbitrary Banach spaces with $T \in \mathscr{B}(Z, W)$ invertible. Then T is bounded below by M if and only if $\|T^{-1}\| \leqslant 1/M$.*

Proof. The following statements are easily seen to be equivalent:

$$\inf_{z \in \partial U(Z)} \|Tz\| \geqslant M > 0 \qquad (M \text{ const})$$

$$\|Tz\| \geqslant M\|z\| \qquad \text{for all} \quad z \in Z$$

$$\|T(T^{-1}(w))\| \geqslant M\|T^{-1}w\| \quad \text{for all} \quad w \in W$$

$$\|w\| \geqslant M\|T^{-1}w\| \qquad \text{for all} \quad w \in W$$

$$\|T^{-1}\| \leqslant 1/M.$$

LEMMA 1.9. *For the map $\imath_m \colon Y_m \to Y$ as defined in Definition 1.5 we have that $\|\imath_m\| \to 1$ as $m \to \infty$.*

Proof. Since $\imath_m^{-1}$ is one-to-one, it is bounded below by Lemma 1.7 and hence has a greatest lower bound. Let $C_m = \operatorname{glb}(\imath_m^{-1})$. By Lemma 1.8 $\|\imath_m\| \leqslant 1/C_m$. Since $\pi_m \mid Y$ is the restriction of a norm 1 operator, $\|\pi_m \mid Y\| \leqslant 1$. This makes $\imath_m$ a norm-increasing map at each point of Y_m so that $1 \leqslant \|\imath_m\| \leqslant 1/C_m$. To prove the lemma it suffices to show that $C_m \to 1$ as $m \to \infty$. Note that $C_m = \inf_{y \in \partial U(Y)} \|(\pi_m \mid Y) y\|$. Since $\|(\pi_m \mid Y) y - y\| \to 0$ as $m \to \infty$, we have $\|\pi_m \mid Y - I \mid Y\| \to 0$ as $m \to \infty$. Therefore, given $\varepsilon > 0$

$$\|(\pi_m \mid Y) y - (I \mid Y) y\| \leqslant \|\pi_m \mid Y - I \mid Y\| < \varepsilon$$

holds uniformly for all y with $\|y\| = 1$, for all m sufficiently large. It follows that $C_m \to 1$ as $m \to \infty$.

LEMMA 1.10. *Referring to Theorem* 1.1, $\|(P\,|\,Y_m)^{-1}\| \to 1$ *as* $m \to \infty$.

Proof. Since $(\pi_m\,|\,Y)$ is the restriction of a norm 1 operator, $\|\pi_m\,|\,Y\| \leqslant 1$. This makes $\imath_m = (\pi_m\,|\,Y)^{-1}$ a norm increasing map at each point of Y_m so that

$$\|(P\,|\,Y_m)^{-1}\| \leqslant \|\imath_m \circ (P\,|\,Y_m)^{-1}\| \leqslant \|\imath_m\|\,\|(P\,|\,Y_m)^{-1}\|. \tag{3}$$

Recalling that $\|\imath_m \circ (P\,|\,Y)^{-1}\|$ and $\|\imath_m\|$ both approach 1 as $m \to \infty$ (Lemmas 1.4 and 1.9, respectively), and letting $m \to \infty$ in the inequalities (3) yields the result.

We proceed now to the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that $(\imath_m \circ P_m \circ \pi_m)\colon X \to Y$ is a projection of X onto Y. Therefore,

$$\|P\| \leqslant \|\imath_m \circ P_m \circ \pi_m\| \leqslant \|\pi_m\|\,\|P_m\|\,\|\imath_m\|.$$

Since $\|\pi_m\| = 1$ and $\|\imath_m\| \to 1$ as $m \to \infty$, it follows that

$$\|P\| \leqslant \underline{\lim}\|P_m\|. \tag{4}$$

We now wish to establish that $\overline{\lim}\,\|P_m\| \leqslant \|P\|$. For this, let $\tilde{P}_m\colon X \to Y_m$ be a minimal projection. Note that $(P\,|\,Y_m)^{-1} \circ P$ is a projection of X onto Y_m. By Lemma 1.10 $\|(P\,|\,Y_m)^{-1}\| \to 1$, as $m \to \infty$. Therefore,

$$\|P_m\| \leqslant \|\tilde{P}_m\,|\,X_m\| \leqslant \|\tilde{P}_m\| \leqslant \|(P\,|\,Y_m)^{-} \circ P\| \leqslant \|P\|\,\|(P\,|\,Y_m)^{-1}\| \tag{5}$$

where the first and third inequalities follow from the fact that P_m and $\tilde{P}_m$ are minimal projections and the second inequality follows because $\tilde{P}_m\,|\,X_m$ is a restriction. Letting $m \to \infty$ in (5) we get

$$\overline{\lim}\,\|P_m\| \leqslant \|P\|. \tag{6}$$

Combining (4) and (6) yields

$$\|P\| \leqslant \underline{\lim}\|P_m\| \leqslant \overline{\lim}\,\|P_m\| \leqslant \|P\|,$$

so that $\|P_m\| \to \|P\|$ as $m \to \infty$. From this it follows that $\|\imath_m \circ P_m \circ \pi_m\| \to \|P\|$ because

$$\|P\| \leqslant \|\imath_m \circ P_m \circ \pi_m\| \leqslant \|\imath_m\|\,\|P_m\|\,\|\pi_m\| \to \|P\|.$$

This completes the proof of part (a).

For the proof of part (b), a new topology is introduced for $\mathscr{B}(X, Y)$, where X and Y are as in Theorem 1.1. Since Y is finite-dimensional it is a dual space, say $Z^* = Y$. Define the "weak*-operator topology" on

$\mathscr{B}(X, Z^*)$ by spcifying that any net of operators $T_\alpha \in \mathscr{B}(X, Z^*)$ converges to an operator $T \in \mathscr{B}(X, Z^*)$ if and only if

$$(T_\alpha x, z) \to (Tx, z) \qquad \text{for all} \quad x \in X,\ z \in Z.$$

We will refer to the weak*-operator topology as simply the τ-topology. It is well known (for example, see Morris and Cheney [15] or Blatter and Cheney [2]) that any subset of $\mathscr{B}(X, Z^*)$ which is norm-bounded and τ-closed is τ-compact. (A short discussion of other aspects of this topology can be found in Holmes [3].)

Let $Q_m = \imath_m \circ P_m \circ \pi_m$. By part (a) there is a constant M such that $\|Q_m\| \leqslant M$ for $m \in N$. Define $F = \{Q \in \mathscr{P}(X, Y) \colon \|Q\| \leqslant M\}$. It is easy to show that F is τ-closed so that F is τ-compact. Therefore, the sequence $\{Q_m\}$ has a cluster point, say Q_0, in F with some subnet of $\{Q_m\}$, say $\{Q_{m_\alpha}\}$, converging to Q_0. Since Z is reflexive, the fact that $|(Q_{m_\alpha}x)z - (Q_0 x)z| \to 0$ for all x in X and all z in Z implies that $|z^{**}(Q_{m_\alpha}x) - z^{**}(Q_0 x)| \to 0$ for all x in X and z^{**} in Z^{**}. That is, $\{Q_{m_\alpha}x\}$ converges in the weak topology on Y to $Q_0 x$ for each x in X. Since Y is finite dimensional $\{Q_{m_\alpha}x\}$ converges to $Q_0 x$ for every x in X in the norm topology on Y. Therefore, Q_0 is a cluster point of the sequence $\{Q_m\}$ in the topology of pointwise convergence on F. Since $\|Q_0\| \leqslant \underline{\lim}\|Q_{m_\alpha}\|$, Q_0 is a minimal projection of X onto Y.

It should perhaps be remarked that $\{Q_m\}$ has a cluster point in F also follows from Theorem 1, chapter 7 of [14].

If X is separable, then since F is an equicontinuous family on X and the set $F(x)$ has compact closure in Y for each x in X, by the Ascoli theorem there is a subsequence of $\{Q_m\}$ which converges pointwise to a continuous operator Q_0. It is easy to see that Q_0 must be a minimal projection of X onto Y.

COROLLARY 1.11. *Let X and Y be as in Theorem* 1.1. *If X possesses a monotone Schauder basis, then the hypothesis of Theorem* 1.1 *are met by letting π_m be the natural projection of X onto the span of the first m basis elements.*

Proof. Let $\{b_i \colon i \in N\}$ be the basis elements and let $\{\alpha_i \colon i \in N\}$ be the corresponding coordinate functionals. It is well known that if the basis $\{b_i\}$ is monotone, then the natural projections, $\pi_m(x) = \sum_{i=1}^{m} \alpha_i(x) b_i$ are each norm one projections onto $\pi_m(X)$. A little more generally, we could let $\{k_m \colon m \in N\}$ be an increasing sequence of positive integers and define $\pi_m(x) = \sum_{i=1}^{k_m} \alpha_i(x) b_i$.

Some spaces with a monotone basis are $C[a, b]$, $L^p[a, b]$ and l^p for $1 \leqslant p < \infty$, and c_0. These spaces are discussed in the next section in conjunction with applications of Theorem 1.1.

Note that if Y has finite co-dimension and a co-minimal projection P_c from X onto Y exists, then $I-P_c$ is a minimal projection from X onto the null space of P_c. If the maps π_m: $X \to X_m$ exist, then Theorem 1.1 is applicable to $I-P_c$.

The problem of finding an analogue of Theorem 1.1 for co-minimal projections of X onto Y when X is infinite dimensional and Y is finite dimensional is still open. Part of the difficulty is that $I-P_c$ has infinite-dimensional range in this case.

2. Applications

In this section we show how Theorem 1.1 can be used to calculate a numerical approximation to P. For this, it is convenient to have some stock examples of spaces with a monotone basis. All of these can be found in Singer [18].

EXAMPLE 2.1. Let $X=C[0,1]$, the real-valued continuous functions on $[0,1]$. Define $x_0(t)\equiv 1$, $x_1(t)=t$,

$$\begin{aligned} x_{2^k+l}(t) &= 0 \qquad \text{for} \quad t \notin \left(\frac{2l-2}{2^{k+1}}, \frac{2l}{2^{k+1}}\right) \\ &= 1 \qquad \text{for} \quad t = \frac{2l-1}{2^{k+1}} \\ &\qquad \text{linear in} \left[\frac{2l-2}{2^{k+1}}, \frac{2l-1}{2^{k+1}}\right] \\ &\qquad \text{and} \qquad \left[\frac{2l-1}{2^{k+1}}, \frac{2l}{2^{k+1}}\right] \end{aligned}$$

($l=1,2,\ldots,2^k$; $k=0,1,2,3,\ldots$).

From x_2 on the basis may be thought of as a collection of "rooftop" functions where the kth level consists of 2^k rooftop functions, where the lth function has support on

$$\left[\frac{2l-2}{2^{k+1}}, \frac{2l}{2^{k+1}}\right].$$

For any $f\in C[0,1]$ we have

$$f(t)=\sum_{i=0}^{\infty} \alpha_i(f)\, x_i(t)$$

where the α_i are the coordinate functionals. If, for $m \in N$ we let $k_m = 2^{m-1} + 1$ and define $(\pi_m f)(t) = \sum_{i=0}^{k_m} \alpha_i(f)\, x_i(t)$; then $\pi_m f$ is the piecewise-linear continuous function interpolating f at the points $t_s = (s-1)/2^{m-1}$, $s = 1, 2,..., k_m$. Since the basis is monotone, $\|\pi_m\| = 1$ for all m, and since the x_i are a basis $\pi_m f \to f$ as $m \to \infty$.

EXAMPLE 2.2. Let X be real $L^p[0, 1]$, $1 \leqslant p < \infty$. Define the *normalized Haar basis* in $L^p[0, 1]$ by

$$Z_1^{(p)}(t) \equiv 1$$

$$\begin{aligned} Z_{2^k+l}^{(p)} &= \sqrt[p]{2^k} && \text{for} \quad t \in \left[\frac{2l-2}{2^{k+1}}, \frac{2l-1}{2^{k+1}}\right] \\ &= -\sqrt[p]{2^k} && \text{for} \quad t \in \left[\frac{2l-1}{2^{k+1}}, \frac{2l}{2^{k+1}}\right] \\ &= 0 && \text{for the other } t \end{aligned}$$

$(l = 1, 2,..., 2^k;\ k = 0, 1, 2,...)$.

For each $x \in L^p[0, 1]$ we may write

$$x(t) = \sum_{i=1}^{\infty} h_i(x)\, Z_i^{(p)}(t)$$

where $h_i(x) = \int_0^1 x(s)\, Z_i^{(q)}(s)\, ds$ for $i \in N$ and $1/p + 1/q = 1$. For each $m = 1, 2,...$ let $k_m = 2^{m-1}$ and define

$$(\pi_m x)(t) = \sum_{i=1}^{k_m} h_i(x)\, Z_i^{(p)}(t).$$

Now each $\pi_m x$ is a step function on equally spaced "steps" of length $1/2^m$.

EXAMPLE 2.3. Let X be one of the real or complex sequence spaces l^p $(1 \leqslant p < \infty)$, or c_0—the space of sequences convergent to zero in the sup norm. Let $\xi = (\xi_1, \xi_2,...)$ be a general element of X. In each case define $\pi_m(\xi) = (\xi_1,..., \xi_m)$. Then π_m is a norm 1 projection onto its range and $\pi_m \xi \to \xi$ as $m \to \infty$ for all $\xi \in X$.

We now show how Theorem 1.1 can be used to calculate a numerical approximation to P. By Theorem 1.1, m can be chosen sufficiently large so that the norm of the projection $\iota_m \circ P_m \circ \pi_m \colon X \to Y$ is close to $\|P\|$. Since ι_m and π_m are known, the problem is to calculate P_m. To illustrate the method, let X in Theorem 1.1 be $C[0, 1]$ and let π_m be as in Example 2.1. Then $X_m (= \pi_m(X))$ and $Y_m (= \pi_m(Y))$ are both Banach spaces with the norm inherited from X. Now X_m consists of piecewise-linear functions with nodes at the points, say $t_1,\dots, t_k$ of $[0, 1]$. Therefore, any $f \in X_m$ may be identified with the vector $(f(t_1), f(t_2),\dots, f(t_k))^T$. Defining $\|f\|_\infty = \max_{1<i<k} |f(t_i)|$ we note that $\|f\|_X = \|f\|_\infty$. Thus the correspondence

$$\pi_m f \Leftrightarrow (f(t_1), f(t_2),\dots, f(t_k))^T$$

is an isometric isomorphism of the Banach space X_m onto $(R^k, \|\cdot\|_\infty)$. Under this correspondence, Y_m is an n-dimensional subspace of R^k, call it Y_n^k. The problem interpreted in this "new" space is to compute a minimal projection P_m: $R^k \to Y_n^k$, when R^k is endowed with the infinity norm, $\|\cdot\|_\infty$. It is well known that this norm induces the norm on matrix operators given by the "maximum absolute row sum" when the elements of R^k are expressed in the standard orthonormal basis. Thus, the problem is to determine a matrix operator, say $M_n = (a_{ij})$, which is a projection from R^k onto Y_n^k, whose maximum absolute row sum, $\max_{1 \leqslant i \leqslant k}\{\sum_{j=1}^k |a_{ij}|\}$, is less than or equal to that of any other matrix which is a projection of R^k onto Y_n^k. A numerical algorithm for solving the minimal (or co-minimal) matrix projection problem when the operator norm is known is developed in [16, Chap. 2].

The same strategy is followed when $X = L^p[0, 1]$, $1 \leqslant p < \infty$. Here each $\pi_m x$ is a step function, say $\pi_m x = \sum_{i=1}^k c_i \chi_{E_i}$, where the E_i are k subintervals of $[0, 1]$ of equal length, say Δ. Now

$$\|\pi_m x\|_p = \left(\int_0^1 |(\pi_m x)(t)|^p \, dt\right)^{1/p} = \left(\sum_{i=1}^k \Delta |c_i|^p\right)^{1/p}$$
$$= \Delta^{1/p} \left(\sum_{i=1}^k |c_i|^p\right)^{1/p} = \Delta^{1/p} \|(c_1, c_2,\dots, c_k)^T\|_p.$$

Thus the correspondence

$$\pi_m x = \sum_{i=1}^k c_i \chi_{E_i} \Leftrightarrow (c_1, c_2,\dots, c_k)^T$$

is an isometric isomorphism of the Banach space X_m (with the norm inherited from X) onto $(R^k, \Delta^{1/p}\|\cdot\|_p)$. The scale factor $\Delta^{1/p}$ plays no part in determining the projection of least norm from R^k onto Y^k because it cancels out in the definition of the operator norm. Thus, we can perform the minimization in the more standard space $(R^k, \|\cdot\|_p)$. Unfortunately, the induced matrix norm is unknown at this time except for the cases $p = 1, 2$, and ∞. If $p = 1$, the induced matrix norm in the standard orthonormal basis is the "maximal absolute column sum," and if $p = \infty$, it is the "maximum absolute row sum." In these two cases, however, the numerical work can be carried out completely. Some examples for these two cases are given in the Appendix.

Obviously, there is a limit beyond which it becomes impractical to calculate P_m by numerical methods. It would be more fruitful to combine the information gained from numerical techniques with some form for the minimal projection. Recently, Chalmers [4, 5], and Chalmers and Metcalf [3] have obtained very general results concerning the structure of minimal projections from $L(Q)$, $1 \leqslant p \leqslant \infty$, onto an n-dimensional subspace Y, when Q is a compact T_1 space. The structure is expressed in terms of equations in which appear (in a non-linear fashion) on the order of n^2 unknown constants. Thus, it is envisioned that it may be possible to use the numerical computation of P_m (for sufficiently large m) to obtain approximate starting values for the constants, after which a Newton's iteration on the defining equations would yield an accurate numerical value for the constants.

3. Appendix

A computer program was written to demonstrate the feasibility of numerically computing P_m. After making the isometric isomorphic identification explained in the last section, the problem that remains is to find a minimal projection, say P, of $X = (R^{n+k}, \|\cdot\|)$ onto a proper subspace Y of dimension n. The algorithm used to numerically compute P is proved in [16]. Briefly, the method is based on the fact that $P(A)$ is the matrix (in the standard orthonormal basis) of a projection of X onto Y if and only if

$$P(A) = VQ(A)V^{-1},$$

where

$$Q(A) = \left[\begin{array}{c|c} I_n & A \\ \hline 0 & 0 \end{array}\right],$$

I_n is the $n \times n$ identity matrix, A is an $n \times k$ matrix and V is a fixed matrix whose first n columns are a basis of Y (expressed in the standard orthonormal basis), with the remaining columns chosen so that V is non-singular. Then $P = P(A_0)$, where A_0 is a matrix which minimizes $\|P(A)\|$, where $\|\cdot\|$ is the induced operator norm explained in the last section. Similarly, a co-minimal projection, say P_c, of X onto Y may be computed from the equation.

$$\|I - P_c\| = \min_A \|VQ_c(A)V^{-1}\|,$$

where

$$Q_c(A) = \left[\begin{array}{c|c} 0 & A \\ \hline 0 & I_k \end{array}\right].$$

Finally, it should be pointed out that the algorithm used to find the minimal and co-minimal projections in Examples 3.2 and 3.3 had converged to at least five decimal places in all cases. In the process of converting from decimals to fractions, these matrices have become exact.

EXAMPLE 3.1. In [9] Franchetti and Cheney computed the minimal projection, P, of $L^1[-1, 1]$ onto $[1, x]$ (the span of 1 and x). They found that $\|P\| = 1.22040\ldots$. Choosing 10 equally spaced points in the interval $[-1, 1]$ and computing a minimal projection on these 10 points by the method explained above yields a projection whose norm is $\|P\| = 1.22302$. The convex minimization was over $n \times k = 2 \times 8$ real variables.

EXAMPLE 3.2. The points $Q = \{-1, -1/3, 1/3, 1\}$ in the interval $[-1, 1]$ were chosen and the minimal projection, P, onto $[1, x, x^2] \mid Q$ in the l^∞ norm was computed. The result is

$$P = \frac{1}{20}\begin{bmatrix} 19 & 3 & -3 & 1 \\ 3 & 11 & 9 & -3 \\ -3 & 9 & 11 & 3 \\ 1 & -3 & 3 & 19 \end{bmatrix}$$

$\|P\|_{l^\infty} = 1.3$

The co-minimal projection, P_c, was found to be

$$P_c = \frac{1}{8}\begin{bmatrix} 7 & 3 & -3 & 1 \\ 1 & 5 & 3 & -1 \\ -1 & 3 & 5 & 1 \\ 1 & -3 & 3 & 7 \end{bmatrix} \qquad (1)$$

$\|P_c\|_{l^\infty} = 14/8 = 1.75$. Note that the matrix $I - P_c$ is given by

$$I - P_c = \frac{1}{8}\begin{bmatrix} 1 & -3 & 3 & -1 \\ -1 & 3 & -3 & 1 \\ 1 & -3 & 3 & -1 \\ 1 & 3 & -3 & 1 \end{bmatrix},$$

and $\|I-P_c\|_{l^\infty}=1.0$. It is well known that $\|I-P\|=1$ if and only if P is a linear best approximation operator. Therefore, the matrix at (1) is a linear best approximation operator in the standard orthonormal basis.

Remark. The fact that P_c is a linear best approximation operator in Examples 3.2 and 3.3 could have been predicted (see Chalmers [6], Hallauer [11], Holmes [12], Price and Cheney [17]) since $|Q|=\dim Y+1$.

EXAMPLE 3.3. The points $Q=\{-3/4, -1/4, 1/4, 3/4\}$ in the interval $[-1, 1]$ were chosen and the minimal projection $P\colon l^1(Q)\to[1, x, x^2]\,|\,Q$ was computed. The result is the same matrix as in example 3.2 (!) but with a different co-minimal projection,

$$P=\frac{1}{20}\begin{bmatrix} 19 & 3 & -3 & 1 \\ 3 & 11 & 9 & -3 \\ -3 & 9 & 11 & 3 \\ 1 & -3 & 3 & 19 \end{bmatrix},$$

$\|P\|_l=1.3$. The co-minimal projection is

$$P_c=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/6 & 1/2 & 1/2 & -1/6 \\ -1/6 & 1/2 & 1/2 & 1/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\|P_c\|_{l^1}=4/3$. Note that $\|I-P_c\|_{l^1}=1.0$ so that P_c is also a linear best approximation operator.

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